The Interval of Convergence for a Power Series – Examples

To review the process:

How to Test a Power Series for Convergence
1. **Find the interval where the series converges absolutely**. We have to use the Ratio or Root Test unless you have a geometric series, in which case you simply use |r| <1 to get the interval of convergence. 2. If the interval of convergence is finite, also **check the endpoints.** Use a Comparison Test, the Integral Test, or the Alternating Series Theorem.

Let's go through some examples:

A: Find the interval of convergence for the series $\sum_{n=0}^{\infty}(x+5)^n$.

There are two ways to do this problem. Method one, if you recognize our series as being geometric, saves you a lot of time!

Method One:

Geometric series : $r = x+5$ Interval of convergence: |r| < 1 -1 < x + 5 < 1 -6 < x < -4	First, identify the series as geometric and show the ratio. We do not have to check the endpoints because we know that the convergence of a geometric series is only defined when |r| < 1. A geometric series is not convergent when r = 1. We are done!!

Method Two:

If you don't recognize the series as being geometric, we have to do the problem the long way. First we will use the ratio test to find the interval of absolute convergence, and then we will find and simplify the endpoint series and test each of them to find out if the series will converge at the endpoints. More work, but it will get the job done.

ratio of successive terms $\frac{(x+5)^{n+1}}{(x+5)^n}=(x+5)$ limit of the ratio: $\lim_{n\to\infty}(x+5)=x+5$ the series will converge: | x + 5 | < 1 The interval of absolute convergence is | x + 5| < 1 -1 < x + 5 < 1 -6 < x < -4	If you don't recognize that you have a geometric series, you will use another method. Here I use the ratio test to find an interval of convergence. The variable here is n, not x, so be careful when you find the limit. Remember that the ratio test states that if the absolute value of the limit of the ratio is less than 1 the series will converge.
Check the endpoint series: at x = -6: $\sum_{n=0}^{\infty}(-6+5)^n=\sum_{n=0}^{\infty}(-1)^n=1-1+1-...$ a divergent series by the n^{th} term test at x = -4: $\sum_{n=0}^{\infty}(-4+5)^n=\sum_{n=0}^{\infty}(1)^n=1+1+1+...$ a divergent series by the n^{th} term test	If we have not identified the series as geometric, we must check the endpoints to see if we would get a convergent series. Put the value of x at each endpoint into the original series and then check the resulting series for convergence. Make sure you **label each endpoint series** to show what endpoint you are testing! In this case our series is not convergent at either endpoint.
interval of convergence: -6 < x < -4	Summarize the information.

B: Find the interval of convergence for the series $\sum_{n=0}^{\infty}(2x)^n$.

Again, we have a geometric series. If we recognize that, we will have much less work to do!
Note: you cannot use method one unless you have identified the series as geometric in your proof!

Method One: Geometric series : $r = 2x$ Interval of convergence: |r| < 1 -1 < 2x < 1 - ½ < x < ½	The easiest way to do this, is to recognize that we have a geometric series. We do not have to check the endpoints because we know that the convergence of a geometric series is only defined when |r| < 1. A geometric series is not convergent when r = 1.
Method Two: nth root : $\sqrt[n]{(2x)^n} = 2x$ limit of the root: $\lim_{n\to\infty}(2x) = 2x$ the series will converge absolutely for: | 2x | < 1 -1 < x < 1 -1/2 < x < 1/2	Here I use the nth root test to find an interval of convergence. Remember that the root test states that if the absolute value of the limit of the root is less than 1 the series will converge. The interval of absolute convergence is -1/2 < x < 1/2
at x = -1/2: $\sum_{n=0}^{\infty}\left(-\frac{1}{2}*2\right)^n = \sum_{n=0}^{\infty}(-1)^n = 1-1+1-\ldots$ divergent by the n^{th} term test **at x = 1/2:** $\sum_{n=0}^{\infty}\left(\frac{1}{2}*2\right)^n = \sum_{n=0}^{\infty}(1)^n = 1+1+1+\ldots$ divergent by the n^{th} term test	**Since we have not identified this series as geometric, we must check the endpoints.** Put the value of x into the series and then check the resulting series for convergent. In this case our series is not convergent at either endpoint.
interval of convergence: -1/2 < x < 1/2	Summarize the information.

C: Find the interval of convergence for the series $\sum_{n=0}^{\infty}\frac{(2x+3)^{2n+1}}{n!}$.

No geometric series this time, so we only have one method to use.

$\sum_{n=0}^{\infty}\frac{(2x+3)^{2n+1}}{n!}$	This is the original series.
ratio of successive terms $\frac{(2x+3)^{2n+3}}{(n+1)!}*\frac{n!}{(2x+3)^{2n+1}}=\frac{(2x+3)^{2}}{n+1}$ limit of the ratio: $\lim_{n\to\infty}\frac{(2x+3)^{2}}{n+1}=0$ the series will converge for all x	I used the ratio test to find an interval of convergence. Since the limit is a finite number, its convergence is not dependent on the value of x. This time we do not have to check the endpoints of any interval. Yeah!
interval of convergence: -∞ < x < ∞	Summarize the information.

D: Find the interval of convergence for the series $\sum_{n=0}^{\infty}\frac{(-1)^{n}x^{n}}{\sqrt{n^{2}+3}}$.

ratio of successive terms : $\frac{x^{n+1}}{\sqrt{(n^2+2n+1)+3}} * \frac{\sqrt{n^2+3}}{x^n}$ $\frac{x^{n+1}}{\sqrt{n^2+2n+4}} * \frac{\sqrt{n^2+3}}{x^n} = \frac{x\sqrt{n^2+3}}{\sqrt{n^2+2n+4}}$ limit of the ratio: $\lim_{n\to\infty} \frac{x\sqrt{n^2+3}}{\sqrt{n^2+2n+4}} = x$ the series will converge absolutely on: -1 < x < 1	I use the ratio test to find an interval of convergence. Again, be careful when you find the limit; n is the variable here that is approaching infinity. Treat x as a constant. According to the ratio test, if the absolute value of the limit of the ratio is less than 1 the series will converge.
at x = -1: $\sum_{n=0}^{\infty} \frac{(-1)^n(-1)^n}{\sqrt{n^2+3}} = \frac{1}{\sqrt{n^2+3}}$ $\sum_{n=0}^{\infty} \frac{1}{\sqrt{n^2}} = \sum_{n=0}^{\infty} \frac{1}{n}$ is the divergent harmonic series $\lim_{n\to\infty} \frac{\frac{1}{\sqrt{n^2+3}}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{n}{\sqrt{n^2+3}} = 1$ so both series diverge by the Limit Comparison Test. **at x = 1:** $\sum_{n=0}^{\infty} \frac{(-1)^n(1)^n}{\sqrt{n^2+3}} = \frac{(-1)^n}{\sqrt{n^2+3}}$ $\lim_{n\to\infty} \frac{1}{\sqrt{n^2+3}} = 0$ The terms approach zero $\frac{1}{\sqrt{(n+1)^2+3}} < \frac{1}{\sqrt{n^2+3}}$ the terms are decreasing, so the series converges by the Alternating Series Theorem.	Check the endpoints. Put the value of x into the series and then check the resulting series for convergent. For x = -1, I used the Limit Comparison Test, comparing it to the divergent harmonic series . For x = 1, I used the Alternating Series Theorem. The terms approach zero and are decreasing, so the series converges. ***** full work for the endpoint series must be shown!**
interval of convergence: -1 < x ≤ 1	Summarize the information. The series converged at the right-hand endpoint but not at the left.

E: Find the interval of convergence for the series $\sum_{n=0}^{\infty} n^n x^n$.

$\sum_{n=0}^{\infty} n^n x^n$	This is the original series.
n[th] root of terms : $\sqrt[n]{n^n x^n} = nx$ limit of the root: $\lim_{n\to\infty} nx = \infty$ the series will diverge	I used the nth root test to find an interval of convergence. This series will diverge.
interval of convergence: none, the series only converges at its center, c = 0	However, by looking at the series I can see that if I let x = 0 the series will converge. **A series will always converge at its center.**

F: Find the interval of convergence of the series $\sum_{n=0}^{\infty}\left(\frac{x^2-1}{2}\right)^n$ and then find its sum within that interval.

Geometric series: r = $\frac{x^2-1}{2}$ the series converges for |r| < 1: $-1 < \frac{x^2-1}{2} < 1$ $-2 < x^2 - 1 < 2$ $-1 < x^2 < 3$, this must be further restricted to $0 < x^2 < 3$ $-\sqrt{3} < x < \sqrt{3}$	I could use the n[th] root test to find an interval of convergence, but it saves a few steps if I recognize this as a geometric series. Remember that if the absolute value of the limit of the ratio is less than 1 the series will converge.
the interval of convergence is $-\sqrt{3} < x < \sqrt{3}$	Since it is geometric, we don't have to check the endpoint series.
$S = \frac{1}{1-\frac{x^2-1}{2}} = \frac{2}{2-\left(x^2-1\right)}$ $S = \frac{2}{3-x^2}$	Now we find the sum. It is a geometric series with a = 1 and r = $(x^2 - 1)/2$

**** Note** – They gave us a hint that this was geometric. The only series that we know how to get the sum of are geometric and telescoping series.